

# Optimal Berry-Esseen bounds on the Poisson space

Ehsan Azmoodeh

Unité de Recherche en Mathématiques, Luxembourg University

`ehsan.azmoodeh@uni.lu`

Giovanni Peccati

Unité de Recherche en Mathématiques, Luxembourg University

`giovanni.peccati@gmail.com`

May 13, 2015

## Abstract

We establish new lower bounds for the normal approximation in the Wasserstein distance of random variables that are functionals of a Poisson measure. Our results generalize previous findings by Nourdin and Peccati (2012, 2015) and Biermé, Bonami, Nourdin and Peccati (2013), involving random variables living on a Gaussian space. Applications are given to optimal Berry-Esseen bounds for edge counting in random geometric graphs.

**Keywords:** Berry-Esseen Bounds; Limit Theorems; Optimal Rates; Poisson Space; Random Graphs; Stein's Method;  $U$ -statistics.

**MSC 2010:** 60H07; 60F05; 05C80.

## 1 Introduction

### 1.1 Overview

Let  $(Z, \mathcal{Z})$  be a Borel space endowed with a  $\sigma$ -finite non-atomic measure  $\mu$ , and let  $\hat{\eta}$  be a compensated Poisson random measure on the state space  $(Z, \mathcal{Z})$ , with non-atomic and  $\sigma$ -finite control measure  $\mu$  (for the rest of the paper, we assume that all random objects are defined on a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ). Consider a sequence of centered random variables  $F_n = F_n(\hat{\eta})$ ,  $n \geq 1$  and assume that, as  $n \rightarrow \infty$ ,  $\text{Var}(F_n) \rightarrow 1$  and  $F_n$  converges in distribution to a standard Gaussian random variable. In recent years (see e.g. [2, 6, 7, 8, 10, 14, 19, 21]) several new techniques – based on the interaction between Stein's method [4] and Malliavin calculus [9] – have been introduced, allowing one to find explicit Berry-Esseen bounds of the type

$$d(F_n, N) \leq \varphi(n), \quad n \geq 1, \quad (1.1)$$

where  $d$  is some appropriate distance between the laws of  $F_n$  and  $N$ , and  $\{\varphi(n) : n \geq 1\}$  is an explicit and strictly positive numerical sequence converging to 0. The aim of this paper is to find some general sufficient conditions, ensuring that the rate of convergence induced by  $\varphi(n)$  in (1.1) is *optimal*, whenever  $d$  equals the 1-Wasserstein distance  $d_W$ , that is:

$$d(F_n, N) = d_W(F_n, N) := \sup_{h \in \text{Lip}(1)} |\mathbb{E}[h(F_n)] - \mathbb{E}[h(N)]|, \quad (1.2)$$

with  $\text{Lip}(a)$  indicating the set of  $a$ -Lipschitz mappings on  $\mathbb{R}$  ( $a > 0$ ). As usual, the rate of convergence induced by  $\varphi(n)$  is said to be optimal if there exists a constant  $c \in (0, 1)$  (independent of  $n$ ) such that, for  $n$  large enough,

$$\frac{d_W(F_n, N)}{\varphi(n)} \in (c, 1]. \quad (1.3)$$

As demonstrated below, our findings generalize to the framework of random point measures some previous findings (see [1, 3, 12, 13]) for random variables living on a Gaussian space. Several important differences between the Poisson and the Gaussian settings will be highlighted as our analysis unfolds. Important new applications  $U$ -statistics, in particular to edge-counting in random geometric graphs, are discussed in Section 4.

## 1.2 Main abstract result (and some preliminaries)

Let the above assumptions and notation prevail. The following elements are needed for the subsequent discussion, and will be formally introduced and discussed in Section 2.2:

- For every  $z \in Z$  and any functional  $F = F(\hat{\eta})$ , the *difference* (or *add-one cost operator*)  $D_z F(\hat{\eta}) = F(\hat{\eta} + \delta_z) - F(\hat{\eta})$ . For reasons that are clarified below, we shall write  $F \in \text{dom } D$ , whenever  $\mathbb{E} \int_Z (D_s F)^2 \mu(ds) < \infty$ .
- The symbol  $L^{-1}$  denotes the *pseudo-inverse of the generator of the Ornstein-Uhlenbeck semigroup* on the Poisson space.

We also denote by  $N \sim \mathcal{N}(0, 1)$  a standard Gaussian random variable with mean zero and variance one. It will be also necessary to consider the family

$$\mathcal{F}_W := \{f : \mathbb{R} \rightarrow \mathbb{R} : \|f'\|_\infty \leq 1 \text{ and } f' \in \text{Lip}(2)\},$$

whereas the notation  $\mathcal{F}_0$  indicates the subset of  $\mathcal{F}_W$  that is composed of twice continuously differentiable functions such that  $\|f'\|_\infty \leq 1$  and  $\|f''\|_\infty \leq 2$ .

For any two sequences  $\{a_n\}_{n \geq 1}$  and  $\{b_n\}_{n \geq 1}$  of non-negative real numbers, the notation  $a_n \sim b_n$  indicates that  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$ .

The next theorem is the main theoretical achievement of the present paper.

**Theorem 1.1.** *Let  $\{F_n : n \geq 1\}$  be a sequence of square-integrable functionals of  $\hat{\eta}$ , such that  $\mathbb{E}(F_n) = 0$ , and  $F_n \in \text{dom } D$ . Let  $\{\varphi(n) : n \geq 1\}$  be a numerical sequence such that  $\varphi(n) \geq \varphi_1(n) + \varphi_2(n)$ , where*

$$\varphi_1(n) := \sqrt{\mathbb{E} \left( 1 - \langle DF_n, -DL^{-1}F_n \rangle_{L^2(\mu)} \right)^2}, \quad (1.4)$$

$$\varphi_2(n) := \mathbb{E} \int_Z (D_z F_n)^2 \times |D_z L^{-1}F_n| \mu(dz). \quad (1.5)$$

(I) *For every  $n$ , one has the estimate  $d_W(F_n, N) \leq \varphi(n)$ .*

(II) *Fix  $f \in \mathcal{F}_0$ , set  $R_n^f(z) := \int_0^1 f''(F_n + (1-u)D_z F_n)u \, du$  for any  $z \in Z$ , and assume moreover that the following asymptotic conditions are in order :*

- (i) (a)  $\varphi(n)$  is finite for every  $n$ ; (b)  $\varphi(n) \rightarrow 0$  as  $n \rightarrow \infty$ ; and (c) there exists  $m \geq 1$  such that  $\varphi(n) > 0$  for all  $n \geq m$ .
- (ii) *For  $\mu(dz)$ -almost every  $z \in Z$ , the sequence  $D_z F_n$  converges in probability towards zero.*

- (iii) *There exist a centered two dimensional Gaussian random vector  $(N_1, N_2)$  with  $\mathbb{E}(N_1^2) = \mathbb{E}(N_2^2) = 1$ , and  $\mathbb{E}(N_1 \times N_2) = \rho$ , and moreover a real number  $\alpha \geq 0$  such that*

$$\left( F_n, \frac{1 - \langle DF_n, -DL^{-1}F_n \rangle_{L^2(\mu)}}{\varphi(n)} \right) \xrightarrow{\text{law}} (N_1, \alpha N_2).$$

- (iv) *There exists a sequence  $\{u_n : n \geq 1\}$  of deterministic and non-negative measurable functions such that  $\int_Z u_n(z) \mu(dz) / \varphi(n) \rightarrow \beta < \infty$ , and moreover*

$$\frac{1}{\varphi(n)} \left\{ \int_Z (D_z F_n)^2 \times (-D_z L^{-1} F_n) \times R_n^f(z) \mu(dz) - \int_Z u_n(z) \times R_n^f(z) \mu(dz) \right\} \xrightarrow{L^1(\Omega)} 0,$$

$$\text{and } \sup_n \varphi(n)^{-(1+\epsilon)} \int_Z u_n(z)^{1+\epsilon} \mu(dz) < \infty, \text{ for some } \epsilon > 0.$$

Then, as  $n \rightarrow \infty$ , we have

$$\frac{\mathbb{E}(f'(F_n) - F_n f(F_n))}{\varphi(n)} \rightarrow \left\{ \frac{\beta}{2} + \rho \alpha \right\} \mathbb{E}(f''(N)).$$

**(III)** *If Assumptions (i)–(iv) at Point **(II)** are verified and  $\rho \alpha \neq \frac{\beta}{2}$ , then the rate of convergence induced by  $\varphi(n)$  is optimal, in the sense of (1.3).*

**Remark 1.1.** It is interesting to observe that Assumptions **(II)**-(ii) and **(II)**-(iv) in the statement of Theorem 1.1 do not have any counterpart in the results on Wiener space obtained in [12]. To see this, let  $X$  denote a isonormal Gaussian process over a real separable Hilbert space  $\mathfrak{H}$ , and assume that  $\{F_n : n \geq 1\}$  is a sequence of smooth functionals (in the sense of Malliavin differentiability) of  $X$  — for example, each element  $F_n$  is a finite sum of multiple Wiener integrals. Assume that  $\mathbb{E}(F_n^2) = 1$ , and write

$$\varphi(n) := \sqrt{\mathbb{E}(1 - \langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}})^2}.$$

Assume that  $\varphi(n) > 0$  for all  $n$  and also that, as  $n \rightarrow \infty$ ,  $\varphi(n) \rightarrow 0$  and the two dimensional random vector  $\left( F_n, \frac{1 - \langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}}}{\varphi(n)} \right)$  converges in distribution to a centered two dimensional Gaussian vector  $(N_1, N_2)$ , such that  $\mathbb{E}(N_1^2) = \mathbb{E}(N_2^2) = 1$  and  $\mathbb{E}(N_1 N_2) = \rho \neq 0$ . Then, the results of [12] imply that, for any function  $f \in \mathcal{F}_W$ ,

$$\frac{\mathbb{E}(f'(F_n) - F_n f(F_n))}{\varphi(n)} \rightarrow \rho \mathbb{E}(f''(N)), \quad (1.6)$$

where, as before,  $N \sim \mathcal{N}(0, 1)$ . This implies in particular that the sequence  $\varphi(n)$  determines an optimal rate of convergence, in the sense of (1.3). Also, on a Gaussian space one has that relation (1.6) extends to functions of the type  $f_x$ , where  $f_x$  is the solution of the Stein's equation associated with the indicator function  $\mathbf{1}_{\{\cdot \leq x\}}$  (see Section 2.3 below):

in this case the limiting value equals  $\frac{\rho}{3}(x^2 - 1) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ .

## 2 Preliminaries

### 2.1 Poisson measures and chaos

As before,  $(Z, \mathcal{Z}, \mu)$  indicates a Borel measure space such that  $Z$  is a Borel space and  $\mu$  is a  $\sigma$ -finite and non-atomic Borel measure. We define the class  $\mathcal{Z}_\mu$  as  $\mathcal{Z}_\mu = \{B \in \mathcal{Z} : \mu(B) < \infty\}$ . The symbol  $\hat{\eta} = \{\hat{\eta}(B) : B \in \mathcal{Z}_\mu\}$  indicates a *compensated Poisson random measure* on  $(Z, \mathcal{Z})$  with control  $\mu$ . This means that  $\hat{\eta}$  is a collection of random variables defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , indexed by the elements of  $\mathcal{Z}_\mu$ , and such that: (i) for

every  $B, C \in \mathcal{Z}_\mu$  such that  $B \cap C = \emptyset$ ,  $\hat{\eta}(B)$  and  $\hat{\eta}(C)$  are independent, (ii) for every  $B \in \mathcal{Z}_\mu$ ,  $\hat{\eta}(B)$  has a centered Poisson distribution with parameter  $\mu(B)$ . Note that properties (i)-(ii) imply, in particular, that  $\hat{\eta}$  is an *independently scattered* (or *completely random*) measure. Without loss of generality, we may assume that  $\mathcal{F} = \sigma(\hat{\eta})$ , and write  $L^2(\mathbb{P}) := L^2(\Omega, \mathcal{F}, \mathbb{P})$ . See e.g. [9, 15] for details on the notions evoked above.

Fix  $n \geq 1$ . We denote by  $L^2(\mu^n)$  the space of real valued functions on  $Z^n$  that are square-integrable with respect to  $\mu^n$ , and we write  $L_s^2(\mu^n)$  to indicate the subspace of  $L^2(\mu^n)$  composed of symmetric functions. We also write  $L^2(\mu) = L^2(\mu^1) = L_s^2(\mu^1)$ . For every  $f \in L_s^2(\mu^n)$ , we denote by  $I_n(f)$  the *multiple Wiener-Itô integral* of order  $n$ , of  $f$  with respect to  $\hat{\eta}$ . Observe that, for every  $m, n \geq 1$ ,  $f \in L_s^2(\mu^n)$  and  $g \in L_s^2(\mu^m)$ , one has the isometric formula (see e.g. [15]):

$$\mathbb{E}[I_n(f)I_m(g)] = n! \langle f, g \rangle_{L^2(\mu^n)} \mathbf{1}_{(n=m)}. \quad (2.1)$$

The Hilbert space of random variables of the type  $I_n(f)$ , where  $n \geq 1$  and  $f \in L_s^2(\mu^n)$  is called the  $n$ th *Wiener chaos* associated with  $\hat{\eta}$ . We also use the following standard notation:  $I_1(f) = \hat{\eta}(f)$ ,  $f \in L^2(\mu)$ ;  $I_0(c) = c$ ,  $c \in \mathbb{R}$ . The following proposition, whose content is known as the *chaotic representation property* of  $\hat{\eta}$ , is one of the crucial results used in this paper. See e.g. [15].

**Proposition 2.1** (Chaotic decomposition). *Every random variable  $F \in L^2(\mathcal{F}, \mathbb{P}) = L^2(\mathbb{P})$  admits a (unique) chaotic decomposition of the type*

$$F = \mathbb{E}(F) + \sum_{n \geq 1} I_n(f_n), \quad (2.2)$$

where the series converges in  $L^2$  and, for each  $n \geq 1$ , the kernel  $f_n$  is an element of  $L_s^2(\mu^n)$ .

## 2.2 Malliavin operators

We recall that the space  $L^2(\mathbb{P}; L^2(\mu)) \simeq L^2(\Omega \times Z, \mathcal{F} \otimes \mathcal{Z}, \mathbb{P} \otimes \mu)$  is the space of the measurable random functions  $u : \Omega \times Z \rightarrow \mathbb{R}$  such that

$$\mathbb{E} \left[ \int_Z u_z^2 \mu(dz) \right] < \infty.$$

In what follows, given  $f \in L_s^2(\mu^q)$  ( $q \geq 2$ ) and  $z \in Z$ , we write  $f(z, \cdot)$  to indicate the function on  $Z^{q-1}$  given by  $(z_1, \dots, z_{q-1}) \rightarrow f(z, z_1, \dots, z_{q-1})$ .

(a) *The derivative operator  $D$ .* The derivative operator, denoted by  $D$ , transforms random variables into random functions. Formally, the domain of  $D$ , written  $\text{dom} D$ , is the set of those random variables  $F \in L^2(\mathbb{P})$  admitting a chaotic decomposition (2.2) such that

$$\sum_{n \geq 1} n n! \|f_n\|_{L^2(\mu^n)}^2 < \infty. \quad (2.3)$$

If  $F$  verifies (2.3) (that is, if  $F \in \text{dom} D$ ), then the random function  $z \rightarrow D_z F$  is given by

$$D_z F = \sum_{n \geq 1} n I_{n-1}(f(z, \cdot)), \quad z \in Z. \quad (2.4)$$

Plainly  $DF \in L^2(\mathbb{P}; L^2(\mu))$ , for every  $F \in \text{dom} D$ . For every random variable of the form  $F = F(\hat{\eta})$  and for every  $z \in Z$ , we write  $F_z = F_z(\hat{\eta}) = F(\hat{\eta} + \delta_z)$ . The following fundamental result combines classic findings from [11] and [16, Lemma 3.1].

**Lemma 2.1.** *For every  $F \in L^2(\mathbb{P})$  one has that  $F$  is in  $\text{dom } D$  if and only if the mapping  $z \mapsto (F_z - F)$  is an element of  $L^2(\mathbb{P}; L^2(\mu))$ . Moreover, in this case one has that  $D_z F = F_z - F$  almost everywhere  $d\mu \otimes d\mathbb{P}$ .*

(b) *The Ornstein-Uhlenbeck generator  $L$ .* The domain of the Ornstein-Uhlenbeck generator (see e.g. [9, 11]), written  $\text{dom } L$ , is given by those  $F \in L^2(\mathbb{P})$  such that their chaotic expansion (2.2) verifies

$$\sum_{n \geq 1} n^2 n! \|f_n\|_{L^2(\mu^n)}^2 < \infty.$$

If  $F \in \text{dom } L$ , then the random variable  $LF$  is given by

$$LF = - \sum_{n \geq 1} n I_n(f_n). \quad (2.5)$$

We will also write  $L^{-1}$  to denote the pseudo-inverse of  $L$ . Note that  $\mathbb{E}(LF) = 0$ , by definition. The following result is a direct consequence of the definitions of  $D$  and  $L$ , and involves the adjoint  $\delta$  of  $D$  (with respect to the space  $L^2(\mathbb{P}; L^2(\mu))$  — see e.g. [9, 11] for a proof).

**Lemma 2.2.** *For every  $F \in \text{dom } L$ , one has that  $F \in \text{dom } D$  and  $DF$  belongs to the domain to the adjoint  $\delta$  of  $D$ . Moreover,*

$$\delta DF = -LF. \quad (2.6)$$

## 2.3 Some estimates based on Stein's method

We shall now present some estimates based on the use of Stein's method for the one-dimensional normal approximation. We refer the reader to the two monographs [4, ?] for a detailed presentation of the subject. Let  $F$  be a random variable and let  $N \sim \mathcal{N}(0, 1)$ , and consider a real-valued function  $h : \mathbb{R} \rightarrow \mathbb{R}$  such that the expectation  $\mathbb{E}[h(X)]$  is well-defined. We recall that the *Stein equation* associated with  $h$  and  $F$  is classically given by

$$h(u) - \mathbb{E}[h(F)] = f'(u) - uf(u), \quad u \in \mathbb{R}. \quad (2.7)$$

A solution to (2.7) is a function  $f$  depending on  $h$  which is Lebesgue a.e.-differentiable, and such that there exists a version of  $f'$  verifying (2.7) for every  $x \in \mathbb{R}$ . The following lemma gathers together some fundamental relations. Recall the notation  $\mathcal{F}_W$  and  $\mathcal{F}_0$  introduced in Section 1.2.

**Lemma 2.3.** (i) *If  $h \in \text{Lip}(1)$ , then (2.7) has a solution  $f_h$  that is an element of  $\mathcal{F}_W$ .*

(ii) *If  $h$  is twice continuously differentiable and  $\|h'\|_\infty, \|h''\|_\infty \leq 1$ , then (2.7) has a solution  $f_h$  that is an element of  $\mathcal{F}_0$ .*

(iii) *Let  $F$  be an integrable random variable. Then,*

$$d_W(F, N) = \sup_{h \in \text{Lip}(1)} |\mathbb{E}[f_h(F)F - f'_h(F)]| \leq \sup_{f \in \mathcal{F}_W} |\mathbb{E}[f(F)F - f'(F)]|.$$

(iv) *If, in addition,  $F$  is a centered element of  $\text{dom } D$ , then*

$$\begin{aligned} d_W(F, N) \leq & \sqrt{\mathbb{E} \left( 1 - \langle DF, -DL^{-1}F \rangle_{L^2(\mu)} \right)^2} \\ & + \mathbb{E} \int_Z (D_z F)^2 \times |D_z L^{-1}F| \mu(dz). \end{aligned} \quad (2.8)$$

Both estimates at Point (i) and (ii) follow e.g. from [5, Theorem 1.1]. Point (iii) is an immediate consequence of the definition of  $d_W$ , as well as of the content of Point (i) and Point (ii) in the statement. Finally, Point (iv) corresponds to the main estimate established in [14].

### 3 Proof of Theorem 1.1

We start with a general lemma.

**Lemma 3.1.** *Let  $F$  be such that  $\mathbb{E}(F) = 0$  and  $F \in \text{dom} D$ . Assume that  $N \sim \mathcal{N}(0, 1)$ . For any  $f \in \mathcal{F}_0$ , and  $z \in Z$  we set*

$$R^f(z) := \int_0^1 f''(F + (1-u)D_z F) u \, du. \quad (3.1)$$

Then,

$$\begin{aligned} \mathbb{E}(f'(F) - Ff(F)) &= \mathbb{E}\left(f'(F) \left(1 - \langle DF, -DL^{-1}F \rangle_{L^2(\mu)}\right)\right) \\ &\quad + \mathbb{E} \int_Z (D_z F)^2 \times (-D_z L^{-1}F) \times R^f(z) \mu(dz). \end{aligned}$$

*Proof.* Using Lemma 2.2 and the characterization of  $\delta$  as the adjoint of  $D$ , we deduce that, for any  $f \in \mathcal{F}_0$

$$\begin{aligned} \mathbb{E}(Ff(F)) &= \mathbb{E}(LL^{-1}Ff(F)) = \mathbb{E}(\delta(-DL^{-1}F)f(F)) \\ &= \mathbb{E}(\langle Df(F), -DL^{-1}F \rangle_{L^2(\mu)}). \end{aligned} \quad (3.2)$$

In view of Lemma 2.1) and of a standard application of Taylor formula, one immediately infers that

$$\begin{aligned} D_z f(F) &:= f(F_z) - f(F) = f'(F)D_z F + \int_F^{F_z} f''(u)(F_z - u) \, du \\ &= f'(F)D_z F + (D_z F)^2 \times \int_0^1 f''(F + (1-u)D_z F) u \, du. \end{aligned} \quad (3.3)$$

Plugging (3.3) into (3.2), we deduce the desired conclusion.  $\square$

*Proof of Theorem 1.1.* Part **(I)** is a direct consequence of Lemma 2.3-(iv). To prove Point **(II)**, we fix  $f \in \mathcal{F}_0$ , and use Lemma 3.1 to deduce that

$$\begin{aligned} \frac{\mathbb{E}(f'(F_n) - F_n f(F_n))}{\varphi(n)} &= \mathbb{E}\left(f'(F_n) \times \frac{1 - \langle DF_n, -DL^{-1}F_n \rangle_{L^2(\mu)}}{\varphi(n)}\right) \\ &\quad + \frac{1}{\varphi(n)} \mathbb{E} \int_Z (D_z F_n)^2 \times (-D_z L^{-1}F_n) \times R_n^f(z) \mu(dz) \\ &:= I_{1,n} + I_{2,n}. \end{aligned}$$

Assumption (iii) implies that

$$\sup_{n \geq 1} \mathbb{E} \left( \frac{1 - \langle DF_n, -DL^{-1}F_n \rangle_{L^2(\mu)}}{\varphi(n)} \right)^2 < +\infty.$$

Since  $\|f'\|_\infty \leq 1$  by assumption, we infer that the class

$$\left\{ f'(F_n) \times \frac{1 - \langle DF_n, -DL^{-1}F_n \rangle_{L^2(\mu)}}{\varphi(n)} : n \geq 1 \right\}$$

is uniformly integrable. Assumption (iii) implies therefore that, as  $n \rightarrow \infty$ ,

$$I_{1,n} \rightarrow \mathbb{E}(f'(N_1) \times \alpha N_2) = \rho \times \alpha \mathbb{E} f''(N).$$

To deal with the term  $I_{2,n}$ , first note that for each  $z \in Z$ , Assumptions (ii) and (iii) and Slutsky Theorem imply that, for any  $u \in (0, 1)$ ,

$$F_n + (1 - u)D_z F_n \xrightarrow{law} N.$$

Therefore, using the fact that  $\|f''\|_\infty \leq 2$ , and by a direct application of the dominated convergence Theorem, we infer that

$$\mathbb{E}R_n^f(z) \rightarrow \int_0^1 \mathbb{E}(f''(N))u du = \frac{1}{2}\mathbb{E}f''(N). \quad (3.4)$$

At this point, Assumption (iv) and the triangle inequality immediately imply that, in order to obtain the desired conclusion, it is sufficient to prove that, as  $n \rightarrow \infty$ ,

$$\frac{1}{\varphi(n)} \int_Z u_n(z) \times \left\{ \mathbb{E}R_n^f(z) - \frac{1}{2}\mathbb{E}f''(N) \right\} \mu(dz) \rightarrow 0. \quad (3.5)$$

To show (3.5), it is enough to prove that the function integrated on the right-hand side is uniformly integrable: this is straightforward, since  $|R_n^f(z) - \frac{1}{2}\mathbb{E}f''(N)| \leq 2$ , and of the fact that the sequence  $n \mapsto \varphi(n)^{-(1+\epsilon)} \int_Z u_n(z)^{1+\epsilon} \mu(dz)$  is bounded. In view of the first equality in Lemma 2.3-(iii), to prove the remaining Point **(III)** in the statement, it is enough to show that there exists a function  $h$  such that  $\|h'\|_\infty, \|h''\|_\infty \leq 1$ , and  $\mathbb{E}[f_h''(N)] \neq 0$ . Selecting  $h(x) = \sin x$ , we deduce from [1, formula (5.2)] that  $\mathbb{E}f_h''(N) = 3^{-1}\mathbb{E}[\sin(N)H_3(N)] = e^{-1/2} > 0$ , thus concluding the proof.  $\square$

## 4 Applications to $U$ -statistics

### 4.1 Preliminaries

In this section, we shall apply our main results to the following situation:

- $\hat{\eta}$  is a compensated Poisson measure on the product space  $(\mathbb{R}_+ \times Z, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z})$  (where  $(Z, \mathcal{Z})$  is a Borel space) with control measure given by

$$\nu := \ell \times \mu, \quad (4.1)$$

with  $\ell(dx) = dx$  equal to the Lebesgue measure and  $\mu$  equal to a  $\sigma$ -finite Borel measure with no atoms.

- For every  $n \geq 1$ , we set  $\hat{\eta}_n$  to be the Poisson measure on  $(Z, \mathcal{Z})$  given by the mapping  $A \mapsto \hat{\eta}_n(A) := \hat{\eta}([0, n] \times A)$  ( $A \in \mathcal{Z}_\mu$ ), in such a way that  $\hat{\eta}_n$  is a Poisson measure on  $(Z, \mathcal{Z})$ , with intensity  $\mu_n := n \times \mu$ .
- For every  $n$ , the random variable  $F_n$  is a  $U$ -statistic of order 2 with respect to the Poisson measure  $\eta_n := \hat{\eta}_n + \mu_n$ , in the sense of the following definition.

**Definition 4.1.** A random variable  $F$  is called a  $U$ -statistic of order 2, based on the Poisson random measure  $\eta_n$  defined above, if there exists a kernel  $h \in L_s^1(\mu^2)$  (that is,  $h$  is symmetric and in  $L_s^1(\mu^2)$ ), such that

$$F = \sum_{(x_1, x_2) \in \eta_{n, \neq}^2} h(x_1, x_2), \quad (4.2)$$

where the symbol  $\eta_{n, \neq}^2$  indicates the class of all 2-dimensional vectors  $(x_1, x_2)$  such that  $x_i$  is in the support of  $\eta_n$  ( $i = 1, 2$ ) and  $x_1 \neq x_2$ .



We recall that, according to the general results proved in [19, Lemma 3.5 and Theorem 3.6], one has that, if a random variable  $F$  as in (4.2) is square-integrable, then necessarily  $h \in L_s^2(\mu^2)$ , and  $F$  admits a representation of the form

$$F = \mathbb{E}(F) + F_1 + F_2 := \mathbb{E}(F) + I_1(h_1) + I_2(h_2), \quad (4.3)$$

where  $I_1$  and  $I_2$  indicate (multiple) Wiener-Itô integrals of order 1 and 2, respectively, with respect to  $\hat{\eta}$ , and

$$\begin{aligned} h_1(t, z) &:= 2\mathbf{1}_{[0,n]}(t) \int_Z h(a, z) \mu_n(da) \\ &= 2\mathbf{1}_{[0,n]}(t) \int_{\mathbb{R}_+ \times Z} \mathbf{1}_{[0,n]}(s) h(a, z) \nu(ds, da) \\ &= 2\mathbf{1}_{[0,n]}(t) n \int_Z h(a, z) \mu(da) \in L^2(\mu), \end{aligned} \quad (4.4)$$

$$h_2((t_1, x_1), (t_2, x_2)) := \mathbf{1}_{[0,n]^2}(t_1, t_2) h(x_1, x_2), \quad (4.5)$$

where  $\nu$  is defined in (4.1).

## 4.2 Edge-counting in random geometric graphs

Let the framework and notation of Section 4.1 prevail, set  $Z = \mathbb{R}^d$ , and assume that  $\mu$  is a probability measure that is absolutely continuous with respect to the Lebesgue measure, with a density  $f$  that is bounded and everywhere continuous. It is a standard result that, in this case, the non-compensated Poisson measure  $\eta_n$  has the same distribution as the point process

$$A \mapsto \sum_{i=1}^{N_n} \delta_{Y_i}(A), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where  $\delta_y$  indicates the Dirac mass at  $y$ ,  $\{Y_i : i \geq 1\}$  is a sequence of i.i.d. random variables with distribution  $\mu$ , and  $N_n$  is an independent Poisson random variable with mean  $n$ . Throughout this section, we consider a sequence  $\{t_n : n \geq 1\}$  of strictly positive numbers decreasing to zero, and consider the sequence of kernels  $\{h_n : n \geq 1\}$  given by

$$h_n : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+ : (x_1, x_2) \mapsto h_n(x_1, x_2) := \frac{1}{2} \mathbf{1}_{\{0 < \|x_1 - x_2\| \leq t_n\}},$$

where, here and for the rest of the section,  $\|\cdot\|$  stands for the Euclidean norm in  $\mathbb{R}^d$ . Then, it is easily seen that, for every  $n$ , the  $U$ -statistic

$$F_n := \sum_{(x_1, x_2) \in \eta_n^2, \neq} h_n(x_1, x_2), \quad (4.6)$$

equals the number of edges in the random geometric graph  $(V_n, E_n)$  where the set of vertices  $V_n$  is given by the points in the support of  $\eta_n$ , and  $\{x, y\} \in E_n$  if and only if  $0 < \|x - y\| \leq t_n$  (in particular, no loops are allowed).

We will now state and prove the main achievement of the section, refining several limit theorems for edge-counting one can find in the literature (see e.g. [2, 7, 8, 18, 19, 20] and the references therein). Observe that, quite remarkably, the conclusion of the forthcoming Theorem 4.1 is independent of the specific form of the density  $f$ . For every  $d$ , we denote by  $\kappa_d$  the volume of the ball with unit radius in  $\mathbb{R}^d$ .

**Theorem 4.1.** *Assume that  $nt_n^d \rightarrow \infty$ , as  $n \rightarrow \infty$ .*

(a) *As  $n \rightarrow \infty$ , one has the exact asymptotics*

$$\mathbb{E}(F_n) \sim \frac{\kappa_d n^2 t_n^d}{2} \int_{\mathbb{R}^d} f(x)^2 dx, \quad \text{Var}(F_n) \sim \frac{\kappa_d^2 n^3 (t_n^d)^2}{4} \int_{\mathbb{R}^d} f(x)^3 dx. \quad (4.7)$$



(b) Define

$$\tilde{F}_n := \frac{F_n - \mathbb{E}F_n}{\sqrt{\text{Var}(F_n)}}, \quad n \geq 1,$$

and let  $N \sim \mathcal{N}(0, 1)$ . Then, there exists a constant  $C \in (0, \infty)$ , independent of  $n$ , such that

$$d_W(\tilde{F}_n, N) \leq \varphi(n) := Cn^{-1/2}. \quad (4.8)$$

(c) If moreover  $n(t_n^d)^3 \rightarrow \infty$ , then there exists a constant  $0 < c < C$  such that, for  $n$  large enough,

$$cn^{-1/2} \leq d_W(\tilde{F}_n, N) \leq Cn^{-1/2},$$

and the rate of convergence induced by the sequence  $\varphi(n) = Cn^{-1/2}$  is therefore optimal.

*Proof.* The two asymptotic results at Point (a) follow from [18, Proposition 3.1] and [18, formula (3.23)], respectively. Point (b) is a special case of the general estimates proved in [8, Theorem 3.3]. In order to prove Point (c) it is therefore sufficient to show that the sequence  $\tilde{F}_n$  verifies Assumptions (ii), (iii) and (iv) of Theorem 1.1-(II), with respect to the control measure  $\nu$  defined in (4.1), and with values of  $\alpha$ ,  $\beta$  and  $\rho$  such that  $\alpha\rho \neq \beta/2$ . First of all, in view of (4.3), one has that, a.e.  $dt \otimes \mu(dz)$ ,

$$D_{t,z}\tilde{F}_n = \frac{1}{\sqrt{\text{Var}(F_n)}} \{h_{1,n}(t, z) + 2I_1(h_{2,n}(t, z, \cdot))\},$$

where the kernels  $h_{1,n}$  and  $h_{2,n}$  are obtained from (4.4) and (4.5), by taking  $h = h_n$ . Since  $h_{1,n}(t, z) = 2\mathbf{1}_{[0,n]}(t)n \int_{\mathbb{R}^d} h(z, a)\mu(da)$ , we obtain that  $\text{Var}(F_n)^{-\frac{1}{2}} \times h_{1,n}(t, z) = O((t_n^{2d})^{-\frac{1}{2}}) \rightarrow 0$ , as  $n \rightarrow \infty$ . Also, using the isometric properties of Poisson multiple integrals,

$$\mathbb{E}\left(\frac{I_1(h_{2,n}((t, z), \cdot))}{\sqrt{\text{Var}(F_n)}}\right)^2 \leq \frac{n \int_{\mathbb{R}^d} h_n(z, x)\mu(dx)}{\text{Var}(F_n)} = O((nt_n^d)^{-2}) \rightarrow 0.$$

It follows that  $D_{t,z}\tilde{F}_n$  converges in probability to zero for  $d\nu$ -almost every  $(t, z) \in \mathbb{R}_+ \times \mathbb{R}^d$ , and Assumption (ii) of Theorem 1.1-(II) is therefore verified. In order to show that Assumption (iii) in Theorem 1.1-(II) also holds, we need to introduce three (standard) auxiliary kernels:

$$\begin{aligned} h_{2,n} \star_2^1 h_{2,n}(t, x) &:= \mathbf{1}_{[0,n]}(t)n \int_{\mathbb{R}^d} h_n^2(x, a)\mu(da) \\ h_{2,n} \star_1^1 h_{2,n}((t, x), (s, y)) &:= \mathbf{1}_{[0,n]}(t)\mathbf{1}_{[0,n]}(s)n \int_{\mathbb{R}^d} h_n(x, a)h_n(y, a)\mu(da) \\ h_{1,n} \star_1^1 h_{2,n}(t, x) &:= 2\mathbf{1}_{[0,n]}(t)n^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_n(a, y)h_n(x, y)\mu(da)\mu(dx). \end{aligned}$$

The following asymptotic relations (for  $n \rightarrow \infty$ ) can be routinely deduced from the calculations contained in [8, Proof of Theorem 3.3] (recall that the symbol ‘ $\sim$ ’ indicates an exact asymptotic relation, and observe moreover that the constant  $C$  is the same appearing in

(4.21)):

$$\|h_{2,n} \star_2^1 h_{2,n}\|_{L^2(\nu)}^2 = O(n^3(t_n^d)^2) \quad (4.9)$$

$$\|h_{2,n} \star_1^1 h_{2,n}\|_{L^2(\nu^2)}^2 = O(n^4(t_n^d)^3) \quad (4.10)$$

$$\|h_{1,n} \star_1^1 h_{2,n}\|_{L^2(\nu)}^2 \sim \frac{\kappa_d^4 n^5 (t_n^d)^4}{4} \int_{\mathbb{R}^d} f(x)^5 dx \quad (4.11)$$

$$\langle h_{1,n}, h_{1,n} \star_1^1 h_{2,n} \rangle_{L^2(\nu)} \sim \frac{\kappa_d^3 n^4 (t_n^d)^3}{2} \int_{\mathbb{R}^d} f(x)^4 dx \quad (4.12)$$

$$\varphi(n) \text{Var}(F_n) \sim C \frac{\kappa_d^2 n^{5/2} (t_n^d)^2}{4} \int_{\mathbb{R}^d} f(x)^3 dx \quad (4.13)$$

$$\|h_{1,n} \star_1^1 h_{2,n}\|_{L^3(\nu)}^3 = O(n^7(t_n^d)^6) \quad (4.14)$$

$$\|h_{1,n}\|_{L^3(\nu)}^3 \sim \kappa_d^3 n^4 (t_n^d)^3 \int_{\mathbb{R}^d} f(x)^4 dx \quad (4.15)$$

$$\|h_{1,n}\|_{L^4(\nu)}^4 = O(n^5(t_n^d)^4) \quad (4.16)$$

$$\|h_{2,n}\|_{L^2(\nu)}^2 = O(n^2(t_n^d)). \quad (4.17)$$

Using the fact that, by definition,  $L^{-1}Y = -q^{-1}Y$  for every random variable  $Y$  living in the  $q$ th Wiener chaos of  $\hat{\eta}$ , we deduce that (using the control measure  $\nu$  defined in (4.1))

$$\begin{aligned} \langle D\tilde{F}_n, -DL^{-1}\tilde{F}_n \rangle_{L^2(\nu)} &= \frac{1}{\text{Var}(F_n)} \left\{ \int_{\mathbb{R}_+ \times \mathbb{R}^d} h_{1,n}^2(t, z) \nu(dt, dz) \right. \\ &\quad \left. + 3 \int_{\mathbb{R}_+ \times \mathbb{R}^d} h_{1,n}(t, z) I_1(h_{2,n}((t, z), \cdot)) \nu(dt, dz) + 2 \int_{\mathbb{R}_+ \times \mathbb{R}^d} I_1^2(h_{2,n}((t, z), \cdot)) \nu(dt, dz) \right\}. \end{aligned}$$

Using a standard multiplication formula for multiple Poisson integrals (see e.g. [15, Section 6.5]) on the of the previous equation, one deduces that

$$\begin{aligned} &\frac{\langle D\tilde{F}_n, -DL^{-1}\tilde{F}_n \rangle_{L^2(\nu)} - 1}{\varphi(n)} \\ &= \frac{1}{\varphi(n) \text{Var}(F_n)} \left\{ 3I_1(h_{1,n} \star_1^1 h_{2,n}) + 2I_1(h_{2,n} \star_2^1 h_{2,n}) + 2I_2(h_{2,n} \star_1^1 h_{2,n}) \right\} \\ &:= X_{1,n} + X_{2,n} + X_{3,n}. \end{aligned}$$

Now, in view of (4.9), (4.10) and (4.13), one has that, as  $n \rightarrow \infty$ ,

$$\mathbb{E}(X_{2,n}^2) = O((nt_n^d)^{-2}) \rightarrow 0, \quad \text{and} \quad \mathbb{E}(X_{3,n}^2) = O((nt_n^d)^{-1}) \rightarrow 0.$$

Also, (4.11) yields that

$$\mathbb{E}(X_{1,n}^2) \rightarrow \frac{9 \int_{\mathbb{R}^d} f(x)^5 dx}{C^2 (\int_{\mathbb{R}^d} f(x)^3 dx)^2} := \alpha^2 > 0.$$

Finally, in view of (4.14) and (4.15),

$$\left\| \frac{h_{1,n} \star_1^1 h_{2,n}}{\varphi(n) \text{Var}(F_n)} \right\|_{L^3(\nu)}^3, \left\| \frac{h_{1,n}}{\varphi(n) \text{Var}(F_n)} \right\|_{L^3(\nu)}^3 = O(n^{-\frac{1}{2}}) \rightarrow 0, \quad \text{and}$$

A standard application of [17, Corollary 3.4] now implies that, as  $n \rightarrow \infty$

$$\left( \frac{I_1(h_{1,n})}{\sqrt{\text{Var}(F_n)}}, -\frac{I_1(3h_{1,n} \star_1^1 h_{2,n})}{\varphi(n) \text{Var}(F_n)} \right) \xrightarrow{\text{law}} (Z_1, Z_2), \quad (4.18)$$

where  $Z_1 \sim \mathcal{N}(0, 1)$  and  $Z_2 \sim \mathcal{N}(0, \alpha^2)$  are two jointly Gaussian random variables such that

$$\rho' := \mathbb{E}(Z_1 Z_2) = -\lim_n \frac{3}{\varphi(n) \text{Var}^{\frac{3}{2}}(F_n)} \langle h_{1,n}, h_{1,n} \star_1^1 h_{2,n} \rangle_{L^2(\nu)} = -\frac{12}{C} \frac{\int_{\mathbb{R}^d} f(x)^4 dx}{(\int_{\mathbb{R}^d} f(x)^3 dx)^{3/2}}.$$

Now, since relation (4.17) implies that  $\text{Var}(F_n)^{-1/2} I_2(h_{2,n})$  converges to zero in probability, we deduce that the sequence

$$\left( \tilde{F}_n, \frac{1 - \langle D\tilde{F}_n, -DL^{-1}\tilde{F}_n \rangle_{L^2(\nu)}}{\varphi(n)} \right), \quad n \geq 1,$$

converges necessarily to the same limit as the one appearing on the RHS of (4.18). We therefore conclude that Assumption (iii) in Theorem 1.1-(II) is verified with  $\alpha$  defined as above, and  $\rho := \rho'/\alpha$ . To conclude the proof, we will now show that Assumption (iv) in Theorem 1.1-(II) is satisfied for

$$u_n = h_{1,n}^3 \times \text{Var}(F_n)^{-3/2} \quad \text{and} \quad \beta = \frac{8}{C} \frac{\int_{\mathbb{R}^d} f(x)^4 dx}{\left( \int_{\mathbb{R}^d} f(x)^3 dx \right)^{3/2}}. \quad (4.19)$$

To see this, we use again a product formula for multiple stochastic integrals to infer that

$$\begin{aligned} & \frac{1}{\varphi(n)} \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}^d} (D_{t,z} \tilde{F}_n)^2 \times (-D_{t,z} L^{-1} \tilde{F}_n) \times R_n^f(t, z) \nu(dt, dz) \\ &= \frac{(\text{Var}(F_n))^{-3/2}}{\varphi(n)} \left\{ \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}^d} h_{1,n}^3(t, z) \times R_n^f(t, z) \nu(dt, dz) \right. \\ &+ 5 \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}^d} h_{1,n}^2(t, z) I_1(h_{2,n}((t, z), \cdot)) \times R_n^f(t, z) \nu(dt, dz) \\ &+ 8 \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}^d} h_{1,n}(t, z) \times I_1(h_{2,n}((t, z), \cdot))^2 \times R_n^f(t, z) \nu(dt, dz) \Big\} \\ &:= B_{1,n} + B_{2,n} + B_{3,n}. \end{aligned}$$

Since relations (4.15) and (4.16) imply that, as  $n \rightarrow \infty$  and using the notation (4.19),

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} \frac{u_n(t, z)}{\varphi(n)} \nu(dt, dz) \rightarrow \beta, \quad \text{and} \quad \int_{\mathbb{R}_+ \times \mathbb{R}^d} \left( \frac{u_n(t, z)}{\varphi(n)} \right)^{4/3} \nu(dt, dz) = O(n^{-1/3}),$$

the proof is concluded once we show that  $\mathbb{E}B_{2,n}, \mathbb{E}B_{3,n} \rightarrow 0$ . In order to deal with  $B_{2,n}$ , we use the fact that  $|R_n^f| \leq 1$ , together with the Cauchy-Schwarz and Jensen inequalities and the isometric properties of multiple integrals, to deduce that

$$\begin{aligned} & \left| \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}^d} h_{1,n}^2(t, z) I_1(h_{2,n}((t, z), \cdot)) \times R_n^f(t, z) \nu(dt, dz) \right| \\ & \leq n^{7/2} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} h_n(z, a) \mu(da) \right)^2 \sqrt{\int_{\mathbb{R}^d} h_n^2(z, a) \mu(da)} \mu(dz) \\ & \leq n^{7/2} \sqrt{\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} h_n(z, a) \mu(da) \right)^2 \left( \int_{\mathbb{R}^d} h_n^2(z, a) \mu(da) \right) \mu(dz)} \\ & = O(n^{7/2} (t_n^d)^{3/2}). \end{aligned}$$

Since we work under the assumption that  $n(t_n^d)^3 \rightarrow \infty$ , this yields that, as  $n \rightarrow \infty$ ,  $\mathbb{E}B_{2,n} = O(n^{-1/2} (t_n^d)^{-3/2}) \rightarrow 0$ . Analogous computations yield that  $\mathbb{E}B_{3,n} = O((n t_n^d)^{-1}) \rightarrow 0$ , and this concludes the proof.  $\square$

### 4.3 Geometric $U$ -statistics of order 2

Our approach is robust enough for extending to more general classes of  $U$ -statistics. In order to see this, let the framework and notation of Section 4.1 prevail, and consider a sequence of *geometric*  $U$ -statistics of order 2 given by

$$F_n := \sum_{(x_1, x_2) \in \eta_{n,\neq}^2} h(x_1, x_2), \quad h \in L^1(\mu^2) \cap L^2(\mu^2). \quad (4.20)$$

A proof similar to that of Theorem 4.1 yields the following statement. (Note that items (a) and (b) in the forthcoming theorem are a consequence of [8, Theorem 7.3], as well [19]).

**Theorem 4.2.** *Assume that the kernels  $h_{1,n}$  and  $h_{2,n}$  are given by the RHS of (4.4) and (4.5).*

- (a) *Let  $h_1(z) := \int_Z h(x, z) \mu(dx)$ , for every  $z \in Z$ . If  $\|h_1\|_{L^2(\mu)} > 0$ , then as  $n \rightarrow \infty$ , one has the exact asymptotic*

$$\text{Var}(F_n) \sim \text{Var}(F_{1,n}) \sim \|h_1\|_{L^2(\mu)}^2 n^3.$$

- (b) *Define*

$$\tilde{F}_n := \frac{F_n - \mathbb{E}F_n}{\sqrt{\text{Var}(F_n)}}, \quad n \geq 1.$$

*Set  $\tilde{h}_{1,n} = (\text{Var}(F_{1,n}))^{-\frac{1}{2}} h_{1,n}$ , and  $\tilde{\varphi}(n) := \|\tilde{h}_{1,n}\|_{L^3(\mu_n)}^3 = \|h_1\|_{L^3(\mu)}^3 \times \|h_1\|_{L^2(\mu)}^{-3} \times n^{-\frac{1}{2}}$ . Let  $N \sim \mathcal{N}(0, 1)$ . Then, there exists a constant  $C \in (0, \infty)$ , independent of  $n$ , such that*

$$d_W(\tilde{F}_n, N) \leq C \tilde{\varphi}(n). \quad (4.21)$$

- (c) *Let  $h_2(x_1, x_2) := h(x_1, x_2)$ , and denote*

$$\alpha_{h_1, h_2}^2 := \frac{9 \|h_1\|_{L^2(\mu)}^2 \|h_1 \star_1^1 h_2\|_{L^2(\mu)}^2}{\|h_1\|_{L^3(\mu)}^6}, \quad \text{and} \quad (4.22)$$

$$\rho_{h_1, h_2} := -\frac{3 \langle h_1, h_1 \star_1^1 h_2 \rangle_{L^2(\mu)}}{\|h_1\|_{L^3(\mu)}^3}. \quad (4.23)$$

*If moreover  $h(x_1, x_2) \geq 0$ , for  $(x_1, x_2) \in Z^2$  a.e.  $\mu^2$ , and also  $\alpha_{h_1, h_2} \times \rho_{h_1, h_2} \neq -1/2$ , then there exists a constant  $0 < c < C$  such that, for  $n$  large enough,*

$$c \tilde{\varphi}(n) \leq d_W(\tilde{F}_n, N) \leq C \tilde{\varphi}(n),$$

*and the rate of convergence induced by the sequence  $\varphi(n) := C \tilde{\varphi}(n)$  is therefore optimal.*

## References

- [1] BIERMÉ, H., BONAMI, A., NOURDIN, I., PECCATI, G. (2013) Optimal Berry-Esseen rates on the Wiener space: the barrier of third and fourth cumulants. *ALEA Lat. Am. J. Probab. Math. Stat.* 9, no. 2, 473-500.
- [2] BOURGUIN, S. AND PECCATI, G. (2014) Portmanteau inequalities on the Poisson space: mixed regimes and multidimensional clustering. *Electron. J. Probab.* 19, no. 66, 1-42.
- [3] CAMPESE, S. (2013) Optimal convergence rates and one-term edge-worth expansions for multidimensional functionals of Gaussian fields. *ALEA Lat. Am. J. Probab. Math. Stat.* 10, no. 2, 881-919.
- [4] CHEN, L. H. Y., GOLDSTEIN, L., SHAO, QI-MAN (2011) *Normal approximation by Stein's method*. Probability and its Applications. Springer.
- [5] DALY (2008) Upper Bounds for Stein-Type Operators. *Electron. J. Probab.* 13, no. 20, 566-587.
- [6] EICHELSBACHER, P., THÄLE, C. (2014) New Berry-Esseen bounds for non-linear functionals of Poisson random measures. *Electron. J. Probab.* 19, no. 102, 1-25.

- [7] LACHÉZE-REY, R., PECCATI, G. (2013) Fine Gaussian fluctuations on the Poisson space, I: contractions, cumulants and geometric random graphs. *Electron. J. Probab.* 18, no. 32, 32 pp.
- [8] LACHÉZE-REY, R., PECCATI, G. (2013) Gaussian fluctuations on the Poisson space II: rescaled kernels, marked processes and geometric U-statistics. *Stochastic Process. Appl.* 123, no. 12, 4186-4218.
- [9] LAST, G. (2014) Stochastic analysis for Poisson processes. Preprint.
- [10] LAST, G., PECCATI, G. AND SCHULTE, M. (2014) Normal approximations on the Poisson space: Mehler's formula, second order Poincaré inequalities and stabilization. Preprint.
- [11] LAST, G., PENROSE, M. (2011) Poisson process Fock space representation, chaos expansion and covariance inequalities. *Probab. Theory Related Fields.* 150, no. 3-4, 663-690.
- [12] NOURDIN, I., PECCATI, G. (2009) Stein's method and exact Berry-Esseen asymptotics for functionals of Gaussian fields. *Ann. Probab.* 37, no. 6, 2231-2261.
- [13] NOURDIN, I., PECCATI, G. (2015) The optimal fourth moment theorem. *Proc. Amer. Math. Soc.* 143, no. 7, 3123-3133.
- [14] PECCATI, G., SOLÉ, J. L., TAQQU, M. S., UTZET, F. (2010) Stein's method and normal approximation of Poisson functionals. *Ann. Probab.* 38, no. 2, 443-478.
- [15] PECCATI, G., TAQQU, M. S. (2010) *Wiener Chaos: Moments, Cumulants and Diagrams*. Springer-Verlag
- [16] PECCATI, G., THAELE, CH. (2013) Gamma limits and U-statistics on the Poisson space *ALEA* 10, no. 1, 525-560.
- [17] PECCATI, G., ZHENG, C. (2010) Multi-dimensional Gaussian fluctuations on the Poisson space. *Electron. J. Probab.* 15, no. 48, 1487-1527.
- [18] PENROSE, M. (2003) *Random Geometric Graphs* Oxford University Press.
- [19] REITZNER, M., SCHULTE, M. (2013) Central limit theorems for U-statistics of Poisson point processes. *Ann. Probab.* 41, no. 6, 3879-3909.
- [20] REITZNER, M., SCHULTE, M., THAELE, CH. (2014) Limit theory for the Gilbert graph. Preprint.
- [21] SCHULTE, M. (2014) Normal approximation of Poisson functionals in Kolmogorov distance. *J. Theoret. Probab.*, to appear.